On the center of a compact group

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February 1, 2008

Abstract

We prove a conjecture due to Baumgärtel and Lledó [1] according to which for every compact group G one has $\widehat{Z(G)} \cong C(G)$, where the 'chain group' $C(G)$ is the free abelian group (written multiplicatively) generated by the set $\widehat{G}$ of isomorphism classes of irreducible representations of G modulo the relations $[Z] = [X] \cdot [Y]$ whenever Z is contained in $X \otimes Y$. Thus the center $Z(G)$ depends only on the (ordered) representation ring of G. Furthermore, we prove that every 't-map' $\varphi : \widehat{G} \to A$ into an abelian group, i.e. every map satisfying $\varphi(Z) = \varphi(X)\varphi(Y)$ whenever $X, Y, Z \in \widehat{G}$ and $Z \prec X \otimes Y$, factors through the restriction map $\widehat{G} \to \widehat{Z(G)}$. All these results generalize to pro-reductive groups over algebraically closed fields of characteristic zero.

1 Introduction

With every compact group G one can associate two canonical compact abelian groups, to wit the center $Z(G)$ and the abelianization $G_{ab} = G/\overline{[G,G]}$. Since every compact group can be recovered from its (abstract) category of finite dimensional unitary representations [3], it is natural to ask whether the said abelian groups can be recovered directly from $\mathrm{Rep}\, G$ without appealing to a reconstruction theorem à la Tannaka-Krein-Doplicher-Roberts or Saavedra-Rivano-Deligne-Milne. Since $\mathrm{Rep}\, G$ is a discrete structure it is clear that one will rather recover the duals $\widehat{G_{ab}}$ and $\widehat{Z(G)}$. In the case of $\widehat{G_{ab}}$ it is well known how to proceed: Writing $\mathcal{C} = \mathrm{Rep}\, G$, let $\mathcal{C}_1 \subset \mathcal{C}$ be the full subcategory of one dimensional representations. Then the set of isomorphism classes of objects in $\mathcal{C}_1$ is a (discrete) abelian group, and it is easy to see that

*Supported by NWO through the 'Pioneer' project no. 616.062.384 of N.P. Landsman.

it is isomorphic to $\widehat{G_{ab}}$. It is natural to ask whether also $\widehat{Z(G)}$ can be recovered directly from $\mathrm{Rep}\, G$.

Motivated by certain operator algebraic considerations closely related to and inspired by [3], Baumgärtel and Lledó [1, Section 5] defined, for every compact group G, a discrete abelian group $C(G)$ in terms of the representation category $\mathrm{Rep}\, G$. They identified a surjective homomorphism $C(G) \to \widehat{Z(G)}$ and conjectured the latter to be an isomorphism. They substantiated this conjecture by explicit verification for several finite and compact Lie groups. (According to [1], the case of $SU(N)$ was checked by C. Schweigert.) In this paper we prove $\widehat{Z(G)} \cong C(G)$ for all compact groups, exploiting a remark made in [4], and we derive two useful corollaries. Despite our general proof the examples in [1] remain quite instructive.

2 Definitions and Preparations

Throughout the paper, G denotes any compact group and $\widehat{G}$ the set of equivalence classes of irreducible representations. We allow ourselves the usual harmless sloppiness of not always distinguishing between an irreducible representation X and its equivalence class $[X] \in \widehat{G}$. (Thus 'Let $X \in \widehat{G}$' means 'Let $\mathcal{X} \in \widehat{G}$ and let $X \in \mathrm{Rep}\, G$ be simple such that $[X] = \mathcal{X}$'.) While $\widehat{G}$ is a group iff G is abelian, there always is a notion of 'homomorphism' into an abelian group:

2.1 Definition *Let G be a compact group and A an abelian group. A map $\varphi : \widehat{G} \to A$ is called a t-map (tensor product compatible) if we have $\varphi(Z) = \varphi(X)\varphi(Y)$ whenever $X, Y, Z \in \widehat{G}$ and $Z \prec X \otimes Y$.*

2.2 Lemma *If $\varphi : \widehat{G} \to A$ is a t-map then $\varphi(1) = 1$, where the first 1 denotes the trivial representation of G, and $\varphi(\overline{X}) = \varphi(X)^{-1}$ for every $X \in \widehat{G}$.*

Proof. We have $\varphi(1) = \varphi(1 \otimes 1) = \varphi(1)\varphi(1)$, thus $\varphi(1) = 1$. For any $X \in \widehat{G}$, we have $1 \prec X \otimes \overline{X}$, thus $1 = \varphi(1) = \varphi(X)\varphi(\overline{X})$, which proves the second claim. ■

The following proposition is essentially due to [1]:

2.3 Proposition *For every compact group G there is a universal t-map $p_G : \widehat{G} \to C(G)$. (Thus for every t-map $\varphi : \widehat{G} \to A$ there is a unique homomorphism $\beta : C(G) \to A$ of abelian groups such that*

$$\begin{tikzcd} \widehat{G} \arrow[r, "p_G"] \arrow[dr, "\varphi"'] & C(G) \arrow[d, "\beta"] \\ & A \end{tikzcd}$$

commutes.) Here the 'chain group' $C(G)$ is the free abelian group (written multiplicatively) generated by the set $\widehat{G}$ of isomorphism classes of irreducible representations of G modulo the relations $[Z] = [X] \cdot [Y]$ whenever Z is contained in $X \otimes Y$. The obvious map $p_G : \widehat{G} \to C(G)$ is a t-map.

Proof. We clearly must take β to send the generator $[X]$ of $C(G)$ to $\varphi([X])$, proving uniqueness. In view of the definition of a t-map this is is compatible with the relations imposed on $C(G)$, whence existence of β. ■

2.4 Remark 1. The above elegant definition of $C(G)$ is due to J. Bernstein. In [1], $C(G)$ was defined as $\widehat{G}/\sim$, where $\sim$ is the equivalence relation given by $X \sim Y$ whenever there exist $n \in \mathbb{N}$ and $Z_1, \ldots, Z_n \in \widehat{G}$ such that both X and Y are contained in $Z_1 \otimes \cdots \otimes Z_n$. Denoting the $\sim$-equivalence class of X is denoted by $\langle X \rangle$, $C(G)$ is an abelian group w.r.t. the operations $\langle X \rangle \langle Y \rangle = \langle Z \rangle$, where Z is any irrep contained in $X \otimes Y$, and $\langle X \rangle^{-1} = \langle \overline{X} \rangle$. The easy verification of the equivalence of the two definitions is left to the reader.

2. A chain group $C(\mathcal{C})$, in general non-abelian, satisfying an analogous universal property can be defined for any fusion category $\mathcal{C}$, but we need only the case $\mathcal{C} = \mathrm{Rep}\, G$ and write $C(G)$ rather than $C(\mathrm{Rep}\, G)$. □

The following, proven in [1], is the most interesting example of a t-map:

2.5 Proposition *The restriction of irreducible representations of G to the center defines a surjective t-map $r_G : \widehat{G} \to \widehat{Z(G)}$. Thus also the homomorphism of abelian groups $\alpha_G : C(G) \to \widehat{Z(G)}$ arising as above is surjective.*

Proof. If $Z \in \widehat{G}$ and $g \in Z(G)$ then $\pi_Z(g)$ commutes with $\pi_Z(G)$, thus by Schur's lemma we have $\pi_Z(g) = \chi_Z(g) 1_Z$, where $\chi_Z(g) \in U(1)$. Clearly, $g \mapsto \chi_Z(g)$ is a homomorphism, thus $\chi_Z \in \widehat{Z(G)}$. This defines a map $r_G : \widehat{G} \to \widehat{Z(G)}$, which is easily seen to be a t-map. Since $Z(G)$ is a closed subgroup of G, [6, Theorem 27.46] says that for every irreducible representation (thus character) χ of $Z(G)$ there is a unitary representation π of G such that $\chi \prec \pi \restriction Z(G)$. We thus have $r_G([\pi]) = \chi$, thus r_G is surjective. Therefore also the map $\alpha_G : C(G) \to \widehat{Z(G)}$ arising from Proposition 2.3 is surjective. ■

For brevity we denote as fusion categories the semisimple $\mathbb{C}$-linear tensor categories with simple unit and two-sided duals, e.g. the C^*-tensor categories with conjugates, direct sums and subobjects of [3]. (We do not assume finiteness.) All subcategories considered below are

full, monoidal, replete (closed under isomorphisms) and closed under direct sums, subobjects and duals, thus they are themselves fusion categories.

2.6 DEFINITION *Let $\mathcal{C}$ be a fusion category. Then $\mathcal{C}_0$ denotes the full tensor subcategory generated by the simple objects X for which the exists a simple object $Y \in \mathcal{C}$ such that $X \prec Y \otimes \overline{Y}$.*

2.7 REMARK The subcategory $\mathcal{C}_0$ of a fusion category seems to have first been considered by Etingof et al. [4, Section 8.5], where the following fact is remarked in parentheses. The proof might be well known, but we are not aware of a suitable reference. □

2.8 PROPOSITION *Let G be a compact group and $\mathcal{C} = \mathrm{Rep}\, G$. Then the category $\mathcal{C}_0$ coincides with the full subcategory $\mathcal{C}_Z \subset \mathcal{C}$ consisting of those representations that are trivial when restricted to $Z(G)$. Thus $\mathcal{C}_0 \simeq \mathrm{Rep}(G/Z(G))$.*

Proof. If $X, Y \in \widehat{G}$ are simple and $X \prec Y \otimes \overline{Y}$ then the restriction of X to $Z(G)$ is trivial, implying $\mathcal{C}_0 \subset \mathcal{C}_Z$. As to the converse, let $g \in G$ be such that $g \in \ker \pi_X$ for all $X \in C_0$. This holds iff $(\pi_Y \otimes \pi_{\overline{Y}})(g) = \mathbf{1}$ for all simple $Y \in \mathrm{Rep}\, G$. The latter means

$$\pi_Y(g) \otimes \pi_Y(g^{-1})^t = \mathbf{1},$$

which is true iff $\pi_Y(g) \in \mathbb{C}\mathbf{1}_Y$. Now, if $g \in G$ is represented by scalars in all irreps $Y \in \widehat{G}$ then $g \in Z(G)$. (This follows from the fact that the irreducible representations separate the elements of G.) In view of the Galois correspondence of full monoidal subcategories $\mathcal{D} \subset \mathrm{Rep}\, G$ and closed normal subgroups $H \subset G$ given by

$$\begin{aligned} H_{\mathcal{D}} &= \{g \in G \mid \pi_X(g) = \mathrm{id}\ \forall X \in \mathcal{D}\}, \\ \mathrm{Obj}\, \mathcal{D}_H &= \{X \in \mathrm{Rep}\, G \mid \pi_X(g) = \mathrm{id}\ \forall g \in H\}. \end{aligned}$$

we have $H_{\mathcal{C}_0} \subset Z(G) = H_{\mathcal{C}_Z}$, thus $\mathcal{C}_Z \subset \mathcal{C}_0$ and therefore $\mathcal{C}_0 = \mathcal{C}_Z$. ■

2.9 LEMMA *Let G be compact and $\mathcal{C} = \mathrm{Rep}\, G$. For a simple object $X \in \mathcal{C}$ we have $p_G([X]) = 1$ iff $X \in \mathcal{C}_0$.*

Proof. If Z and $X_i, Y_i,\ i = 1, \ldots, n$ are simple with $X_i \prec Y_i \otimes \overline{Y}_i$ and $Z \prec X_1 \otimes \cdots \otimes X_n$ then $1, Z \prec Y_1 \otimes \overline{Y}_1 \otimes \cdots \otimes Y_n \otimes \overline{Y}_n$, thus $Z \sim 1$. This implies that $p_G([X]) = \langle X \rangle = 1$ for every simple $X \in \mathcal{C}_0$. Conversely, let $X \in \mathcal{C}$ be simple such that $p_G([X]) = 1$. This is equivalent to $X \sim 1$, thus there are simple $Y_1, \ldots, Y_n$ such that $1, X \prec Y_1 \otimes \cdots \otimes Y_n$. Then $X \prec Y_1 \otimes \cdots \otimes Y_n \otimes \overline{Y}_1 \otimes \cdots \otimes \overline{Y}_n$, and therefore $X \in \mathcal{C}_0$. ■

3 Results

3.1 THEOREM *The homomorphism* $\alpha_G : C(G) \to \widehat{Z(G)}$ *is an isomorphism for every compact group* G.

Proof. Since all maps in the diagram

$$\begin{tikzcd} \widehat{G} \arrow[r, "p_G"] \arrow[dr, "r_G"'] & C(G) \arrow[d, "\alpha_G"] \\ & \widehat{Z(G)} \end{tikzcd}$$

are surjective, α_G is an isomorphism iff $\ker p_G = \ker r_G$. By Lemma 2.9, $[X] \in \ker p_G$ iff $X \in \mathcal{C}_0$. On the other hand, $[X] \in \ker r_G$ iff $X \in \mathcal{C}_Z$. By Proposition 2.8 we have $\mathcal{C}_0 = \mathcal{C}_Z$, thus we are done. ■

$C(G)$ is defined in terms of the set $\widehat{G}$ and the multiplicities $N_{ij}^k = \dim \mathrm{Hom}(\pi_k, \pi_i \otimes \pi_j),\ i, j, k \in \widehat{G}$ (the 'fusion rules' in physicist parlance). The same information is contained in the representation ring $R(G)$ provided we take its canonical $\mathbb{Z}-$basis or its order structure [5] into account. We thus have the following

3.2 COROLLARY *The center of a compact group* G *depends only on the (ordered) representation ring* $R(G)$*, not on the associativity constraint or the symmetry of the tensor category* $\mathrm{Rep}\, G$. *(In general, both the latter are needed to determine* G *up to isomorphism.)*

3.3 REMARK A considerably stronger result holds for *connected* compact groups: Every isomorphism of the (ordered) representation rings of two such groups is induced by an isomorphism of the groups, cf. [5]. For non-connected groups this is wrong: The finite groups D_{8l} and Q_{8l} are non-isomorphic but have isomorphic representation rings, cf. [5]. Yet, as remarked in [1, Section 5.1], the centers are isomorphic (to $\mathbb{Z}/2\mathbb{Z}$), as they must by Corollary 3.2. □

As an obvious consequence of Proposition 2.3 and Theorem 3.1 we have:

3.4 COROLLARY *Let* G *be a compact group and* A *an abelian group. Then every t-map* $\varphi : \widehat{G} \to A$ *factors through* $\widehat{Z(G)}$*, i.e. there is a homomorphism* $\beta : \widehat{Z(G)} \to A$ *of abelian groups such that*

$$\begin{tikzcd} \widehat{G} \arrow[r, "r_G"] \arrow[dr, "\varphi"'] & \widehat{Z(G)} \arrow[d, "\beta"] \\ & A \end{tikzcd}$$

commutes.

3.5 Remark This result should be considered as dual to the well known (and much easier) fact that every homomorphism $G \to A$ from a group into an abelian group factors through the quotient map $G \to G_{\mathrm{ab}}$. $\square$

3.6 Remark The results of this note were formulated for compact groups mainly because of the author's taste and background. In view of [2] all results of this paper generalize without change to pro-reductive algebraic groups over algebraically closed fields of characteristic zero. $\square$

Acknowledgments. I am grateful to the DFG-Graduiertenkolleg 'Hierarchie und Symmetrie in mathematischen Modellen' for supporting a one week visit to the RWTH Aachen. In particular I thank Fernando Lledó for the invitation, for drawing my attention to [1], many stimulating discussions, comments on the first version of this paper and for the pizza.

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